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Integral Equation Formulation for Transonic Lifting Profiles

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Introduction

THE integral equation method, originated by Oswatitsch[1] for studying the direct problem of steady inviscid irrotational transonic flow past a thin symmetric profile at zero incidence, is now well established. It has been extended and modified by a number of authors, such as Gullstrand et at.,[2] Zierep,[3] Nφrstrud,[4] Niyogi,[5], Nixon and Hancock,[6] Frohn,[7] and others. A controversy originated recently, from a criticism by Nixon[8] regarding the correctness of the integral equation formulation of Nφrstrud[9] for the lifting profile flow problem. According to Nixon, Nφrstrud's formulation leads to a nonunique solution. Nφrstrud has given two, apparently different, integral equation formulations,[4,9] for the transonic lifting profile flow problem. The purpose of the present Note is to show that the formulations of Nixon and Hancock and Nφrstrud[4] are equivalent.

Different Forms of Formulation

The problem under consideration is to study the steady inviscid irrotational transonic flow of a compressible fluid past a thin unsymmetric profile at small incidence, with freestream Mach number $M_\infty < 1$. The transonic small disturbance continuity equation and the irrotationality condition for this case may be reformulated in terms of a system of two-dimensional nonlinear singular integral equations. Three different forms of integral equation formulations for this problem have been given in Refs. 4, 9, and 6, among which the present Note shows that the formulations in Refs. 4 and 6 are equivalent.

According to Nφrstrud,[4] the reduced perturbation potential $\Phi(X,Y)$ of the above problem, satisfies the following system of four equations

$$\Phi_X^+(X,0)=\bar{\Phi}_X^+(X,0) - \frac{1}{2\pi}\int\int_{-\infty}^{\infty}[\Phi_\xi^+\Phi_{\xi\xi}^+ + \Phi_\xi^-\Phi_{\xi\xi}^-]\frac{\partial}{\partial\xi}\left(\ln\frac{1}{R}\right)d\xi\, d\eta \tag{1a}$$

$$\Phi_X^-(X,0)=\bar{\Phi}_X^-(X,0) \tag{1b}$$

$$\Phi_Y^-(X,0)=\bar{\Phi}_Y^-(X,0) - \frac{1}{2\pi}\int\int_{-\infty}^{\infty}[\Phi_\xi^+\Phi_{\xi\xi}^- + \Phi_\xi^-\Phi_{\xi\xi}^+]\frac{\partial}{\partial\eta}\left(\ln\frac{1}{R}\right)d\xi\, d\eta \tag{1c}$$

$$\Phi_Y^+(X,0)=\bar{\Phi}_Y^+(X,0) \tag{1d}$$

where the superscripts + and − denote, respectively, the symmetric and antisymmetric parts defined by

$$\Phi=\Phi^+ + \Phi^-, \qquad \Phi^+(X,Y)=\Phi^+(X,-Y),$$
$$\Phi^-(X,Y)=-\Phi^-(X,-Y) \tag{2}$$

and the subscripts X and Y denote partial differentiation with respect to X and Y, respectively, and

$$R=\sqrt{(X-\xi)^2+(Y-\eta)^2}$$

The overbar on Φ, denotes a solution of the Laplace equation. It is an unknown of the problem to be determined along with the unknown nonlinear solution Φ by means of the boundary conditions and the irrotationality condition. Furthermore, $\bar{\Phi}$ should not be confused with the Prandtl solution Φ_P, which is a known quantity, being the solution in reduced coordinates of the Laplace equation and the same boundary condition as that of the nonlinear problem. The reduced velocity potential Φ is related to the true velocity potential φ by

$$\Phi(X,Y)=K(\varphi-u_\infty x-v_\infty y)/[(1-M_\infty^2)u_\infty] \tag{3a}$$

and the reduced coordinates denoted by the corresponding capital letters by

$$X=x, \qquad Y=y\sqrt{1-M_\infty^2} \tag{3b}$$

and u_∞, v_∞ denote freestream velocity components.

The parameter K is a function of the freestream Mach number M_∞, for which different approximate values may be used. For example, Oswatitsch uses the value

$$K=(1-M_\infty^2)/[(1/M_\infty^*)-1] \tag{4}$$

M_∞^* denoting the critical freestream Mach number, and Spreiter takes

$$K=M_\infty^2(\gamma+1) \tag{5}$$

Other useful values of K are given in Ref. 4. In the above formulation, there are eight unknowns, viz., Φ_X^+, Φ_X^-, Φ_Y^+, Φ_Y^- and $\bar{\Phi}_X^+$, $\bar{\Phi}_X^-$, $\bar{\Phi}_Y^+$, $\bar{\Phi}_Y^-$.

A second integral formulation for the above problem has been given by Nixon and Hancock[6] who obtained the following equations for shock-free flow

$$U(X,+0)+U(X,-0)-\frac{U^2(X,+0)+U^2(X,-0)}{4} = \frac{1}{\pi}\int_0^1\frac{\Delta V(\xi)}{X-\xi}d\xi - \lim_{y\to+0}\frac{1}{4\pi}\int_S\int\psi_{\xi X}(X,\xi;Y,\eta)[U^2(\xi,\eta)+U^2(\xi,-\eta)]dS \tag{6a}$$

and

$$V(X,+0)+V(X,-0)=-\frac{1}{\pi}\int_0^1\frac{U(\xi,+0)-U(\xi,-0)}{X-\xi}d\xi - \lim_{Y\to+0}\frac{1}{4\pi}\int_S\int\psi_{\xi Y}(X,\xi;Y,\eta)[U^2(\xi,\eta)-U^2(\xi,-\eta)]dS \tag{6b}$$

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where

$$\psi(X,\xi;Y,\eta) = \ln[(X-\xi)^2 + (Y-\eta)^2]^{1/2} \tag{7}$$

and S denotes whole space from which the singular point $\xi = X$, $\eta = Y$ has been excluded by means of a circle around it of infinitesimal radius, and the slit $0 \le \xi \le 1$, $\eta = \pm 0$ where the profile is situated, has been excluded.

In the following we show that system (1) is equivalent to systems (6) and (7).

Deduction of the Second Formulation from the First

We start from the Nϕrstrud equation (1). Integrating Eq. (1a) by parts with respect to ξ yields

$$U^+(X,0) = \bar{U}^+(X,0) + \tfrac{1}{4}[\{U^+(X,0)\}^2 + \{U^-(X,0)\}^2] - \frac{1}{2\pi}\int_{\odot}\int_{-\infty}^{\infty} \frac{[U^+(\xi,\eta)]^2 + [U^-(\xi,\eta)]^2}{2}\psi_{\xi X}\,d\xi d\eta \tag{8}$$

where the symbol $\odot$ indicates that the singularity has been removed by a circle of infinitesimal radius around it and taking the limit as the radius approaches zero. Now, by definition,

$$U(X,+0) = U^+ + U^-, \quad U(X,-0) = U^+ - U^- \tag{9a}$$

so that

$$U(X,+0) + U(X,-0) = 2U^+$$

and

$$U^2(X,+0) + U^2(X,-0) = 2[\{U^+(X,0)\}^2 + \{U^-(X,0)\}^2] \tag{9b}$$

$$U^2(\xi,\eta) - U^2(\xi,-\eta) = 4U^+(\xi,\eta)U^-(\xi,\eta) \tag{9c}$$

It consequently follows from Eq. (8), using Eq. (1b), that

$$U(X,+0) = \bar{U}^+(X,0) + \bar{U}^-(X,0) + \tfrac{1}{4}[\{U^+(X,0)\}^2 + \{U^-(X,0)\}^2] - \frac{1}{2\pi}\int_{\odot}\int_{-\infty}^{\infty} \times \frac{\{U^+(\xi,\eta)\}^2 + \{U^-(\xi,\eta)\}^2}{2}\psi_{\xi X}\,d\xi d\eta \tag{10a}$$

and

$$U(X,-0) = U^+(X,0) - U^-(X,0) = \bar{U}^-(X,0)\bar{U}^-(X,0) + \tfrac{1}{4}[\{U^+(X,0)\}^2 + \{U^-(X,0)\}^2] - \frac{1}{2\pi}\int_{\odot}\int_{-\infty}^{\infty} \frac{\{U^+(\xi,\eta)\}^2 + \{U^-(\xi,\eta)\}^2}{2}\psi_{\xi X}\,d\xi d\eta \tag{10b}$$

Adding Eqs. (10) and using Eq. (9b) it follows that

$$U(X,+0) + U(X,-0) = 2\bar{U}^+(X,0) + \tfrac{1}{4}[U^2(X,+0) + U^2(X,-0] - \frac{1}{4\pi}\int_{\odot}\int_{-\infty}^{\infty} [U^2(\xi,\eta) + U^2(\xi,-\eta)]\psi_{\xi X}\,d\xi d\eta \tag{11}$$

which is identical with the first equation of Nixon-Hancock, viz. Eq. (7a), where the line integral is equal to the quantity $2\bar{U}^+(X,0)$, that is

$$\bar{U}^+(X,0) = \frac{1}{2\pi}\int_0^1 \frac{\Delta V(\xi)}{X-\xi}d\xi \tag{12}$$

On the other hand, integrating Eq. (1c) by parts with respect to ξ yields

$$V^-(X,0) = \bar{V}^-(X,0) - \frac{1}{2\pi}\int_{\odot}\int_{-\infty}^{\infty} U^+(\xi,\eta)U^-(\xi,\eta)\psi_{\xi\eta}\,d\xi d\eta \tag{13}$$

By addition and subtraction it follows from Eqs. (13) and (1d) that

$$V^-(X,0) + V^+(X,0) = \bar{V}^+(X,0) + \bar{V}^-(X,0) - \frac{1}{2\pi}\int_{\odot}\int_{-\infty}^{\infty} U^+(\xi,\eta)U^-(\xi,\eta)\psi_{\xi\eta}\,d\xi d\eta \tag{14a}$$

and

$$V^-(X,0) - V^+(X,0) = -\bar{V}^+(X,0) + \bar{V}^-(X,0) - \frac{1}{2\pi}\int_{\odot}\int_{-\infty}^{\infty} U^+(\xi,\eta)U^-(\xi,\eta)\psi_{\xi\eta}\,d\xi d\eta \tag{14b}$$

Noting that

$$V(X,0) = V^+ + V^-, \quad V(X,-0) = -V^+ + V^- \tag{15}$$

it follows from Eqs. (14) by addition, and using Eq. (9c) that

$$V(X,+0) + V(X,-0) = 2\bar{V}^-(X,0) - \frac{1}{4\pi}\int_{\odot}\int_{-\infty}^{\infty} [U^2(\xi,\eta) - U^2(\xi,-\eta)]\psi_{\xi\eta}\,d\xi d\eta \tag{16}$$

which is the second Nixon-Hancock equation (7b), where the line integral has been replaced by the unknown quantity $2\bar{V}^-(X,0)$; that is

$$\bar{V}^-(X,0) = \frac{1}{2\pi}\int_0^1 \frac{U(\xi,+0) - U(\xi,-0)}{X-\xi}d\xi \tag{17}$$

For solving the above systems, the boundary conditions of the problem and the irrotationality condition must be taken into account. The steps taken in the above proof may be retraced to obtain system (1) from that of system (6), showing their equivalence.

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